Density estimates for the exponential functionals of fractional Brownian motion

Nguyễn Thu Hằng

Hanoi University of Mining and Geology

Annual Conference on Probability and Related Topics
15/12/2021

This report is based on the content of the following article

N.T. Dung, N.T. Hang, P.T.P. Thuy, Density estimates for the exponential functionals of fractional Brownian motion. *Comptes Rendus Mathematique*, 2021.

Content

1. Introduction
2. Preliminaries
    - Malliavin calculus
    - Some general results
3. Main results

1. Introduction

Let $B^H = (B_t^H)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. We consider the exponential functional of the form

$$F = \int_0^T e^{as+\sigma B_s^H} ds, \tag{1}$$

where $T > 0$, $a \in \mathbb{R}$ and $\sigma > 0$ are constants.

1. Introduction

Let $B^H = (B^H_t)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. We consider the exponential functional of the form

$$F = \int_0^T e^{as+\sigma B^H_s} ds, \tag{1}$$

where $T > 0, a \in \mathbb{R}$ and $\sigma > 0$ are constants.

- The special case, $H = \frac{1}{2}$, have been well studied, see [6, 7, 10].

1. Introduction

Let $B^H = (B_t^H)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. We consider the exponential functional of the form

$$F = \int_0^T e^{as+\sigma B_s^H} ds, \tag{1}$$

where $T > 0, a \in \mathbb{R}$ and $\sigma > 0$ are constants.

- The special case, $H = \frac{1}{2}$, have been well studied, see [6, 7, 10].
- $H \neq \frac{1}{2}$

1. Introduction

Let $B^H = (B^H_t)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. We consider the exponential functional of the form

$$F = \int_0^T e^{as+\sigma B^H_s} ds, \tag{1}$$

where $T > 0, a \in \mathbb{R}$ and $\sigma > 0$ are constants.

- The special case, $H = \frac{1}{2}$, have been well studied, see [6, 7, 10].
- $H \neq \frac{1}{2}$

Based on the techniques of Malliavin calculus, we estimated for the density function of F.

2. Preliminaries

2.1. Malliavin calculus

In the whole paper, we assume $H > \frac{1}{2}$. Under this assumption, fBm admits the Volterra representation

$$B_t^H = \int_0^t K(t,s)dB_s, \tag{2}$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion and for some normalizing constant c_H, the kernel K is given by

$$K(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \ 0 < s \le t \le T. \tag{3}$$

2. Preliminaries
2.1. Malliavin calculus

$(B_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{F},P)$, $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$. For $h\in L^2[0,T]$, we denote by $B(h)$ the Wiener integral

$$B(h)=\int_0^T h(t)dB_t.$$

2. Preliminaries
2.1. Malliavin calculus

$(B_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. For $h \in L^2[0, T]$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t)dB_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(B(h_1), ..., B(h_n)), \tag{4}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), h_1, ..., h_n \in L^2[0, T]$.

2. Preliminaries

2.1. Malliavin calculus

$(B_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. For $h \in L^2[0, T]$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t)dB_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(B(h_1), ..., B(h_n)), \tag{4}$$

where $n \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^n), h_1, ..., h_n \in L^2[0, T]$.

$$D_t F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(B(h_1), ..., B(h_n))h_k(t). \tag{5}$$

2.1. Malliavin calculus

More generally, for each $k \geq 1$, we can define the iterated derivative operator by setting

$$D^k_{t_1,\dots,t_k} F = D_{t_1} \dots D_{t_k} F. \tag{6}$$

2.1. Malliavin calculus

More generally, for each $k \geq 1$, we can define the iterated derivative operator by setting

$$D^k_{t_1,\ldots,t_k} F = D_{t_1} \ldots D_{t_k} F. \tag{6}$$

For any $p, k \geq 1$, we shall denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|^p_{k,p} := E|F|^p + E\left[\int_0^T |D_{t_1} F|^p dt_1\right] + \ldots + E\left[\int_0^T \ldots \int_0^T |D^k_{t_1,\ldots,t_k} F|^p dt_1 \ldots dt_k\right]. \tag{7}$$

2.1. Malliavin calculus

A random variable F is said to be Malliavin differentiable if it belongs to $\mathbb{D}^{1,2}$. For any $F \in \mathbb{D}^{1,2}$, the Clark-Ocone formula says that

$$F - E[F] = \int_0^T E[D_s F|\mathcal{F}_s]dB_s. \tag{8}$$

2.1. Malliavin calculus

A random variable F is said to be Malliavin differentiable if it belongs to $\mathbb{D}^{1,2}$. For any $F \in \mathbb{D}^{1,2}$, the Clark-Ocone formula says that

$$F - E[F] = \int_0^T E[D_s F|\mathcal{F}_s]dB_s. \tag{8}$$

Moreover, any $F, G \in \mathbb{D}^{1,2}$, we have the following covariance formula

$$\mathrm{Cov}(F, G) = E\left[\int_0^T D_s F E[D_s G|\mathcal{F}_s]ds\right]. \tag{9}$$

2.2. Some general results

In order to obtain the density estimates for exponential functionals we need the following general results.

Proposition 2.1.

Let q, α, β be three positive real numbers such that $\frac{1}{q} + \frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let F be a random variable in the space $\mathbb{D}^{2,\alpha}$, such that $E[||DF||_H^{-2\beta}] < \infty$. Then the density $\rho_F(x)$ of F can be estimated as follows

$$\rho_F(x) \leq c_{q,\alpha,\beta}(P(F \leq x))^{1/q} \times \left(E[||DF||_H^{-1}] + ||D^2F||_{L^\alpha(\Omega; H \otimes H)} \parallel ||DF||_H^{-2} \parallel_\beta \right), x \in \mathbb{R}, \tag{10}$$

where $c_{q,\alpha,\beta}$ is a positive constant and $H = L^2[0, T]$.

2.2. Some general results

Proposition 2.2.

Let $F \in \mathbb{D}^{2,4}$ *be such that* $E[F] = 0$. *Define the random variable*

$$\Phi_F := \int_0^T D_s F E[D_s F|\mathcal{F}_s] ds. \tag{11}$$

Assume that $\Phi_F \neq 0$ *a.s. and the random variables* $\frac{F}{\Phi_F}$ *and* $\frac{1}{\Phi_F^2} \int_0^T D_s \Phi_F E[D_s F|\mathcal{F}_s] ds$ *belong to* $L^2(\Omega)$. *Then the law of* F *has a continuous density given by*

$$\rho_F(x) = \rho_F(0) \exp\left(-\int_0^x h_F(z) dz \right) \exp\left(-\int_0^x w_F(z) dz \right), \ \ x \in \operatorname{supp} \rho_F, \tag{12}$$

where the functions w_F *and* h_F *are defined by*

$$w_F(z) := E\left[\frac{F}{\Phi_F} \Big| F = z \right], \ \ h_F(z) := E\left[\frac{1}{\Phi_F^2} \int_0^T D_s \Phi_F E[D_s F|\mathcal{F}_s] ds \Big| F = z \right]. \tag{13}$$

3. The main results

Our idea is to consider the random variable $X := \ln F - E[\ln F]$ and use the relation

$$\rho_F(x) = \frac{1}{x}\rho_X(\ln x - E[\ln F]), \; x > 0. \tag{14}$$

3. The main results

Our idea is to consider the random variable $X := \ln F - E[\ln F]$ and use the relation

$$\rho_F(x) = \frac{1}{x}\rho_X(\ln x - E[\ln F]),\ \ x > 0. \tag{14}$$

Proposition 3.1.

$$0 \le D_\theta X \le \sigma K(T,\theta)\ a.s. \tag{15}$$

$$0 \le D_r D_\theta X \le 2\sigma^2 K(T,\theta) K(T,r)\ a.s. \tag{16}$$

3. The main results

Lemma 3.1.

Define

$$M_r := E\left[F|\mathcal{F}_r\right] = E\left[\int_0^T e^{as+\sigma B_s^H} ds \Big| \mathcal{F}_r\right], \; 0 \le r \le T.$$

Then, for every $p \ge 2$, *we have*

$$E\left[\left(\max_{0\le r\le T} M_r\right)^p\right] \le C < \infty, \tag{17}$$

where C *is a positive constant depending on* p, T, a, σ *and* H.

3. The main results

Proposition 3.2.

Let X be as in Proposition (3.1). We define $\Phi_X := \int_0^T D_s X E[D_s X|\mathcal{F}_s]ds$. Then,

$$|\Phi_X|^{-1} \in L^p(\Omega),\ \forall\, p \geq 1.$$

We also have

$$\left(\int_0^T |D_\theta X|^2 d\theta\right)^{-1} \in L^p(\Omega),\ \forall\ p \geq 1.$$

3. The main results

We now are in a position to bound the density $\rho_F(x)$ of F. We first use Proposition (2.1) to estimate the left tail of the density.

Theorem 3.1.

$$\rho_F(x) \leq \frac{c}{x} \exp\left(-\frac{(\ln x - E[\ln F])^2}{8\sigma^2 T^{2H}}\right), \quad 0 < x \leq e^{E[\ln F]}, \tag{18}$$

where c is a positive constant.

3. The main results

We now are in a position to bound the density $\rho_F(x)$ of F. We first use Proposition (2.1) to estimate the left tail of the density.

Theorem 3.1.

$$\rho_F(x) \leq \frac{c}{x} \exp\left(-\frac{(\ln x - E[\ln F])^2}{8\sigma^2 T^{2H}}\right), \quad 0 < x \leq e^{E[\ln F]}, \tag{18}$$

where c is a positive constant.

We use Proposition (2.2) to estimate the right tail of the density.

Theorem 3.2.

$$\rho_F(x) \leq \frac{c}{x} \exp\left(-\frac{(\ln x - E[\ln F])^2}{2\sigma^2 T^{2H}}\right), \quad x > e^{E[\ln F]}, \tag{19}$$

where c is a positive constant.

References

[1] M. Besalú, A. Kohatsu-Higa, S. Tindel, Gaussian-type lower bounds for the density of solutions of SDEs driven by fractional Brownian motions. *Ann. Probab.* 44 (2016), no. 1, 399–443.

[2] N.T. Dung, The density of solutions to multifractional stochastic Volterra integro-differential equations. *Nonlinear Anal.* 130 (2016), 176–189.

[3] N.T. Dung, Kolmogorov distance between the exponential functionals of fractional Brownian motion. *C. R. Math. Acad. Sci. Paris* 357 (2019), no. 7, 629–635.

[4] N.T. Dung, Gaussian lower bounds for the density via Malliavin calculus. *C. R. Math. Acad. Sci. Paris* 358 (2020), no. 1, 79–88.

[5] J. Liu, Ciprian A. Tudor, Stochastic heat equation with fractional Laplacian and fractional noise: existence of the solution and analysis of its density. *Acta Math. Sci. Ser. B (Engl. Ed.)* 37 (2017), no. 6, 1545–1566.

References (tiếp theo...)

[6] H. Matsumoto, M. Yor, Exponential functionals of Brownian motion. I. Probability laws at fixed time. *Probab. Surv.* 2 (2005), 312–347.

[7] H. Matsumoto, M. Yor, Exponential functionals of Brownian motion. II. Some related diffusion processes. *Probab. Surv.* 2 (2005), 348–384.

[8] D. Nualart, *The Malliavin calculus and related topics*. Probability and its Applications. Springer-Verlag, Berlin, second edition, 2006.

[9] N. V. Tan, Smoothness and Gaussian density estimates for stochastic functional differential equations with fractional noise. *Stat. Optim. Inf. Comput.* 8 (2020), no. 4, 822–833.

[10] M. Yor, *Exponential functionals of Brownian motion and related processes.* Springer-Verlag, Berlin, 2001.

Thank You!